SYMMETRIES AND REDUCTION OF MULTIPLICATIVE 2-FORMS

HENRIQUE BURSZTYN AND ALEJANDRO CABRERA

Abstract. This paper is concerned with symmetries of closed multiplicative 2-forms on Lie groupoids and their infinitesimal counterparts. We use them to study Lie group actions on Dirac manifolds by Dirac diffeomorphisms and their lifts to presymplectic groupoids, building on recent work of Fernandes-Ortega-Ratiu [11] on Poisson actions.

Dedicated to Tudor Ratiu on the occasion of his 60th birthday

Contents

1. Introduction 1
Acknowledgments 3
2. Preliminaries 3
2.1. Symmetries of vector bundles 3
2.2. The geometric set-up 4
2.3. Dirac structures 5
3. Momentum maps and lifted Hamiltonian actions 6
4. Reduction of closed IM 2-forms 8
5. Global reduction 9
6. The case of Dirac structures 11
6.1. Comparing reductions 11
6.2. A non-integrable quotient 13
7. Relation with the path-space model 16
References 17

1. Introduction

This paper studies symmetries of closed multiplicative 2-forms on Lie groupoids as well as their infinitesimal counterparts. This study leads to an extension of some of the recent work of Fernandes-Ortega-Ratiu [11] on Poisson actions and their Hamiltonian lifts to symplectic groupoids to the framework of Dirac structures.

Consider a Lie group G acting on a Poisson manifold M by Poisson diffeomorphisms, and suppose that $\mathcal{G}$ is the source-simply-connected symplectic groupoid integrating M (see e.g. [7]). A key observation that can be traced back to [17] is that, even when the G-action on M does not admit a momentum map, it can always be lifted to a Hamiltonian G-action on $\mathcal{G}$ with a momentum map $J : \mathcal{G} \to \mathfrak{g}^*$ suitably compatible with the groupoid structure (in the sense of (3.6) below).

In [11], Fernandes, Ortega and Ratiu use this canonical momentum map on $\mathcal{G}$ as a tool for studying actions on M by Poisson diffeomorphisms. For example, when the action is free and proper, so that the quotient M/G is again a Poisson manifold, they show that the Marsden-Weinstein quotient $J^{-1}(0)/G$ is a symplectic groupoid for M/G (though not necessarily the source-simply-connected one). The simplest instance of this result is when M is an arbitrary manifold (equipped with the zero Poisson structure), in which case one just recovers the well-known fact that any G-action on M, when lifted to T^*M, is Hamiltonian with momentum map

$$J_{can} : T^*M \to \mathfrak{g}^*, \quad \langle J_{can}(\alpha), u\rangle = \alpha(u_M), \tag{1.1}$$

where $u \in \mathfrak{g}$ and u_M is the infinitesimal generator for the G-action on M. The reduction of T^*M at level zero is $T^*(M/G)$, which is just the symplectic groupoid of M/G. Our goal in this note is to place some of these results from [11] in the broader context of Dirac structures [8], offering an alternative viewpoint based on methods from [2, 3] and pointing out additional subtleties that arise in this generality.

Our starting point is [5], where the global objects associated with Dirac manifolds are identified; these objects are referred to as *presymplectic groupoids*, and their role in the theory of Dirac structures is entirely analogous to that of symplectic groupoids in Poisson geometry. A presymplectic groupoid is a Lie groupoid equipped with a closed multiplicative 2-form but, in contrast with symplectic groupoids, this 2-form may be degenerate (though in a "controlled" way). Let $(\mathcal{G}, \omega)$ be a source-simply-connected presymplectic groupoid integrating a Dirac structure L on M, and suppose that a Lie group G acts on M by Dirac diffeomorphisms. We observe that, just as in the case of Poisson actions, the action on M lifts to a Hamiltonian action on $\mathcal{G}$, in the sense that there is a G-equivariant map $J : \mathcal{G} \to \mathfrak{g}^*$ satisfying

$$i_{u_{\mathcal{G}}}\omega = -\mathrm{d}\langle J, u\rangle, \quad \forall u \in \mathfrak{g}, \tag{1.2}$$

where $u_{\mathcal{G}}$ is the infinitesimal generator associated with u for the G-action on $\mathcal{G}$. Note, however, that condition (1.2) does not necessarily determine the infinitesimal action since ω may be degenerate.

When the G-action on M is free and proper, the quotient M/G is a smooth manifold that inherits (with the aid of extra regularity conditions to be specified in Section 6) natural geometrical structures: on the one hand, a Marsden-Weinstein type reduction of $\mathcal{G}$ produces a Lie groupoid $\mathcal{G}_{red}$ over M/G equipped with a closed multiplicative 2-form; on the other hand, the G-invariant Dirac structure L on M may be pushed forward to a Dirac structure L_{quot} on M/G. Unlike the case of Poisson manifolds, however, $\mathcal{G}_{red}$ is generally *not* a presymplectic groupoid for $(M/G, L_{red})$ (in the terminology of [5, Sec. 4], $\mathcal{G}_{red}$ will be proven to be an *over*-presymplectic groupoid). As we will discuss, it may happen that the quotient $(M/G, L_{quot})$ does not admit any presymplectic groupoid at all; i.e., L_{quot} may not be integrable as a Lie algebroid, despite the fact that L was. Our approach to Dirac structures and presymplectic groupoids is through the notion of *IM 2-forms* [5] (see also [1, 2, 3]), differently from [11] that uses path spaces [9] (see also [6, 19]). At the end of the paper, we discuss how to reconcile both viewpoints.

The increasing role of Dirac structures in mechanical problems has been one of our motivations to revisit the work in [11] from this broader perspective. Even in the context of Poisson geometry, Dirac structures naturally appear in the stratified geometry of non-free Poisson actions, see Sections 2.3 and 4.3 of [11]. A more

complete picture, that we leave for future work, should include proper actions on Dirac manifolds which are not necessarily free; Jotz, Ratiu and Sniatycki have begun this study in [12, 13]. Another source of inspiration for the present note is the possibility of using Dirac structures as a tool for studying symplectic groupoids of Poisson homogeneous spaces [14], yet to be explored.

The paper is structured as follows: in Section 2, we recall IM 2-forms on Lie algebroids and present the geometric set-up that will be considered in the paper; Section 3 discusses the Hamiltonian properties of actions on Lie groupoids that lift symmetries of closed IM 2-forms; Section 4 describes the reduction of closed IM 2-forms, while Section 5 deals with its global version, i.e., reduction of closed multiplicative 2-forms; the particular situation of Dirac structures is treated in Section 6; in Section 7, we relate the approach in this paper to the viewpoint of path spaces in [11].

It is a great pleasure to dedicate this note to Tudor, whose work has been central in so many aspects of Poisson geometry and geometric mechanics, including many of the issues discussed here. We are thankful to him for his mathematical insights, unmatched enthusiasm, and constant support.

Acknowledgments. We thank Rui L. Fernandes for helpful advice and Juan-Pablo Ortega for encouraging the writing of this note. We also thank the referees for valuable comments. Bursztyn's research was supported by CNPq and Faperj. Cabrera thanks the University of Toronto and IMPA for their hospitality while this work was being completed and CNPq for financial support.

2. Preliminaries

2.1. **Symmetries of vector bundles.** Let M be a smooth manifold. For a smooth vector bundle $E \to M$, $\mathrm{Aut}(E)$ denotes its group of vector-bundle automorphisms. An automorphism of E covers a diffeomorphism of M, so there is a homomorphism $\mathrm{Aut}(E) \to \mathrm{Diff}(M)$. Any $\Phi \in \mathrm{Aut}(E)$, covering $\varphi \in \mathrm{Diff}(M)$, acts on the space of sections of E by

$$\Phi_* : \Gamma(E) \to \Gamma(E), \quad \Phi_*(s) = \Phi \circ s \circ \varphi^{-1}. \tag{2.1}$$

Let G be a Lie group with Lie algebra $\mathfrak{g}$. In this paper, an action of G on E is always assumed to be by vector-bundle automorphisms, i.e., defined by a group homomorphism $G \to \mathrm{Aut}(E)$; any G-action on E naturally covers, and it is said to be a *lift* of, an action of G on M.

The space of infinitesimal automorphisms of E, denoted by $\mathrm{Der}(E)$, is the set of pairs (X, D), where $X \in \mathfrak{X}(M)$ is a vector field on M and $D : \Gamma(E) \to \Gamma(E)$ is a linear endomorphism such that

$$D(fs) = (\mathcal{L}_X f)s + fD(s), \quad \forall f \in C^\infty(M),\ s \in \Gamma(E). \tag{2.2}$$

For a one-parameter family of automorphisms Φ_t in $\mathrm{Aut}(E)$, $\Phi_0 = Id$, the corresponding infinitesimal automorphism is given by $D(s) = \frac{d}{dt}\big|_{t=0}(\Phi_t)_*(s)$. The space $\mathrm{Der}(E)$ has a natural Lie algebra structure, given by commutators, for which the projection $\mathrm{Der}(E) \to \mathfrak{X}(M)$, $(X, D) \mapsto X$, is a Lie algebra homomorphism. Any G-action on E gives rise to a Lie algebra homomorphism $\mathfrak{g} \to \mathrm{Der}(E)$ for which the composition $\mathfrak{g} \to \mathrm{Der}(E) \to \mathfrak{X}(M)$ agrees with the infinitesimal counterpart of the G-action on M covered by the G-action on E.

Suppose G acts on M freely and properly; let $p : M \to M/G$ be the natural projection, which is a surjective submersion. Any lift of this action to a G-action on E determines a unique vector bundle (up to isomorphism) over M/G, denoted by

$$E/G \to M/G, \tag{2.3}$$

such that the pull-back bundle $p^*(E/G)$ is naturally identified with E, and the pull-back of sections identifies $\Gamma(E/G)$ with the space of G-invariant sections $\Gamma(E)^G$.

2.2. **The geometric set-up.** This paper will be concerned with the following geometric objects:

(i) A smooth manifold M acted upon by a Lie group G.
(ii) A Lie algebroid $A \to M$, with Lie bracket $[\cdot,\cdot]$ on $\Gamma(A)$ and anchor map $\rho : A \to TM$, equipped with a closed IM 2-form [5] $\mu : A \to T^*M$.
(iii) An action of G on A by Lie algebroid automorphisms which lifts the G-action on M and for which $\mu : A \to T^*M$ is G-equivariant.

Let us briefly recall the objects that appear in (ii) and (iii).

The notion of *closed IM 2-form* [5] in (ii) refers to a morphism of vector bundles $\mu : A \to T^*M$, covering the identity map, satisfying

$$i_{\rho(a)}\mu(b) = -i_{\rho(b)}\mu(a) \tag{2.4}$$

$$\mu([a,b]) = \mathcal{L}_{\rho(a)}\mu(b) - i_{\rho(b)}d\mu(a) = \mathcal{L}_{\rho(a)}\mu(b) - \mathcal{L}_{\rho(b)}\mu(a) + \mathrm{d}i_{\rho(b)}\mu(a). \tag{2.5}$$

In (iii), we consider the group of *automorphisms* of the Lie algebroid A, i.e., the subgroup of vector-bundle automorphisms $\mathrm{Aut}(A)$ defined by

$$\mathrm{Sym}(A) = \{\Phi : A \to A \in \mathrm{Aut}(A) \,|\, \Phi_*([a,b]) = [\Phi_*(a), \Phi_*(b)]\, \forall\, a,b \in \Gamma(A)\}. \tag{2.6}$$

One can directly verify that any $\Phi \in \mathrm{Sym}(A)$, covering $\varphi \in \mathrm{Diff}(M)$, satisfies

$$\rho \circ \Phi = \mathrm{d}\varphi \circ \rho, \tag{2.7}$$

as a result of the identity $\Phi_*([a, fb]) = [\Phi_*(a), \Phi_*(fb)]$, $\forall f \in C^\infty(M)$, and the Leibniz rule. The infinitesimal Lie-algebroid automorphisms form a Lie subalgebra of $\mathrm{Der}(A)$ given by

$$\mathrm{sym}(A) = \{(X, D) \in \mathrm{Der}(A) \,|\, D([a,b]) = [D(a), b] + [a, D(b)], \forall a, b \in \Gamma(A)\}. \tag{2.8}$$

In analogy with (2.7), any $(X, D) \in \mathrm{sym}(A)$ satisfies

$$\rho(D(a)) = [X, \rho(a)], \quad \forall a \in \Gamma(A); \tag{2.9}$$

this can be verified directly via the identity $D([a, fb]) = [D(a), fb] + [a, D(fb)]$, $\forall f \in C^\infty(M)$, the Leibniz identity and (2.2).

The G-action on A in (iii) is given by a group homomorphism

$$G \to \mathrm{Sym}(A) \subseteq \mathrm{Aut}(A).$$

The equivariance of μ in (iii) is with respect to the canonical lift of the G-action on M to T^*M. The G-action on A gives rise to an infinitesimal $\mathfrak{g}$-action on A, defined by a Lie algebra homomorphism $\mathfrak{g} \to \mathrm{sym}(A) \subseteq \mathrm{Der}(A)$ necessarily of the form

$$u \mapsto (u_M, D), \tag{2.10}$$

where u_M is the infinitesimal generator of the G-action on M defined by $u \in \mathfrak{g}$. Infinitesimally, the equivariance of μ becomes

$$\mathcal{L}_{u_M}\mu(a) = \mu(D(a)), \quad \forall a \in \Gamma(A), \tag{2.11}$$

for all $(u_M, D) \in \mathrm{sym}(A)$ as in (2.10).

A central theme in this paper is transferring the geometric information in (i), (ii), (iii) to a Lie groupoid $\mathcal{G}$ integrating A. Our notation and conventions for a Lie groupoid $\mathcal{G}$ over M are as follows: the source and target maps are denoted by s and t, the set $\mathcal{G}^{(2)} \subset \mathcal{G} \times \mathcal{G}$ of composable pairs (g,h) is defined by the condition $\mathsf{s}(g) = \mathsf{t}(h)$, and the multiplication is denoted by $m : \mathcal{G}^{(2)} \to \mathcal{G}$, $m(g,h) = gh$; the unit map $M \hookrightarrow \mathcal{G}$ is used to identify M with its image in $\mathcal{G}$. The Lie algebroid of $\mathcal{G}$ is $A\mathcal{G} = \ker(\mathrm{d}\mathsf{s})|_M$, with anchor $\mathrm{d}\mathsf{t}|_A : A \to M$ and bracket induced by right-invariant vector fields.

2.3. **Dirac structures.** The main example of the set-up (i), (ii), (iii) that we will have in mind is given as follows. Consider the vector bundle $\mathbb{T}M := TM \oplus T^*M$ over M equipped with the nondegenerate symmetric fibrewise bilinear form given at each $x \in M$ by

$$\langle (X,\alpha), (Y,\beta) \rangle = \beta(X) + \alpha(Y), \quad X, Y \in T_xM,\ \alpha, \beta \in T_x^*M, \tag{2.12}$$

and the Courant bracket $[\![\cdot,\cdot]\!] : \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$,

$$[\![(X,\alpha), (Y,\beta)]\!] = ([X,Y], \mathcal{L}_X\beta - i_Y\mathrm{d}\alpha). \tag{2.13}$$

We denote by $\mathrm{pr}_T : \mathbb{T}M \to TM$ and $\mathrm{pr}_{T^*} : \mathbb{T}M \to T^*M$ the canonical projections.

A *Dirac structure* [8] on M is a vector subbundle $L \subset \mathbb{T}M$ satisfying $L = L^\perp$ with respect to $\langle\cdot,\cdot\rangle$, and which is involutive with respect to $[\![\cdot,\cdot]\!]$, i.e., $[\![\Gamma(L), \Gamma(L)]\!] \subseteq \Gamma(L)$. Any Poisson structure $\pi \in \Gamma(\wedge^2 TM)$ may be viewed as a Dirac structure via its graph, i.e.,

$$L = \{(\pi^\sharp(\alpha), \alpha) \,|\, \alpha \in T^*M\},$$

where $\pi^\sharp : T^*M \to TM$ is given by $\pi^\sharp(\alpha) = i_\alpha \pi$; closed 2-forms may be viewed as Dirac structures analogously, as graphs of the associated bundle maps $TM \to T^*M$.

For a Dirac structure L on M, the vector bundle $L \to M$ inherits a Lie algebroid structure, with bracket on $\Gamma(L)$ given by the restriction of $[\![\cdot,\cdot]\!]$ and anchor given by restriction of pr_T to L:

$$[\cdot,\cdot]_L = [\![\cdot,\cdot]\!]|_{\Gamma(L)}, \quad \rho_L := \mathrm{pr}_T|_L : L \to TM. \tag{2.14}$$

Moreover, the map

$$\mu_L := \mathrm{pr}_{T^*}|_L : L \to T^*M \tag{2.15}$$

is a closed IM 2-form[1].

A diffeomorphism $\varphi : M \to M$ is said to preserve a Dirac structure L if its lift

$$(\mathrm{d}\varphi, (\mathrm{d}\varphi^{-1})^*) : \mathbb{T}M \to \mathbb{T}M \tag{2.16}$$

preserves L; equivalently, the following holds for all $x \in M$:

$$L_{\varphi(x)} = \{(\mathrm{d}\varphi(X), (\mathrm{d}\varphi^{-1})^*(\alpha)) \,|\, (X,\alpha) \in L_x\}.$$

In particular, φ is both a backward and forward Dirac map (see e.g. [5, Sec. 2.1] and references therein).

The situation that will concern us is that of a Lie group G acting, freely and properly, on a manifold M by diffeomorphisms preserving a Dirac structure $L \subset \mathbb{T}M$.

[1]Conversely, given any Lie algebroid A and closed IM 2-form $\mu : A \to T^*M$, we may consider the image of the map $A \to \mathbb{T}M$, $a \mapsto (\rho(a), \mu(a))$; if $\mathrm{rank}(A) = \dim(M)$ and $\ker(\rho) \cap \ker(\mu) = \{0\}$, then this map is an isomorphism from A onto its image, which is then a Dirac structure on M.

This fits into the framework of (i), (ii), (iii) if we set $A = L$, $\mu = \mu_L$ as in (2.15), and the G-action on A to be the natural lift of the G-action on M via (2.16) restricted to L. When M is a Poisson manifold, the Lie algebroid in question is naturally identified with T^*M, in such a way that μ_L is just the identity map; in this case, the information in (i), (ii), (iii) boils down to an action on M by Poisson diffeomorphisms, as considered in [11].

Although we are primarily interested in Dirac structures, it turns out that more general closed IM 2-forms naturally arise when one considers reduction by symmetries; so we will work in the more general context of (i), (ii), (iii) from the outset.

3. Momentum maps and lifted Hamiltonian actions

Let us consider the geometric set-up described in (i), (ii), (iii) of Section 2.2. Let $J_{can} : T^*M \to \mathfrak{g}^*$ be the momentum map of the canonical lifting of the G-action on M to T^*M, given by (1.1), and let

$$J_A := J_{can} \circ \mu : A \to \mathfrak{g}^*. \tag{3.1}$$

Lemma 3.1. *The map J_A is a morphism of Lie algebroids, where $\mathfrak{g}^*$ is viewed as a trivial Lie algebroid over a point (i.e., a Lie algebra with trivial bracket).*

Proof. Note that $J_A : A \to \mathfrak{g}^*$ is a morphism of vector bundles, because both $\mu : A \to T^*M$ and $J_{can} : T^*M \to \mathfrak{g}^*$ are. The remaining condition to be checked is

$$J_A([a, b]) = \mathcal{L}_{\rho(a)} J_A(b) - \mathcal{L}_{\rho(b)} J_A(a), \quad \forall\, a, b \in \Gamma(A), \tag{3.2}$$

as an equality in $C^\infty(M, \mathfrak{g}^*)$.

Pairing $J_A([a, b])$ with $v \in \mathfrak{g}$, and using (1.1) and (2.5), we get

$$\langle J_{can}(\mu([a, b])), v\rangle = \langle \mu([a, b]), v_M\rangle = \big\langle \mathcal{L}_{\rho(a)}\mu(b) - \mathcal{L}_{\rho(b)}\mu(a) + \mathrm{d}i_{\rho(b)}\mu(a), v_M\big\rangle.$$

The right-hand-side of the previous equation agrees with

$$\big\langle \mathcal{L}_{\rho(a)}\mu(b), v_M\big\rangle - \big\langle \mathcal{L}_{\rho(b)}\mu(a), v_M\big\rangle + \mathcal{L}_{v_M} i_{\rho(b)}\mu(a). \tag{3.3}$$

Using (2.4), (2.9), as well as the infinitesimal equivariance of μ (2.11), we obtain

$$\begin{aligned} \mathcal{L}_{v_M} i_{\rho(b)}\mu(a) =& \langle \mathcal{L}_{v_M}\mu(a), \rho(b)\rangle + \langle \mu(a), [v_M, \rho(b)]\rangle \\ =& \langle \mu(D(a)), \rho(b)\rangle + \langle \mu(a), [v_M, \rho(b)]\rangle \\ =& -\langle \mu(b), \rho(D(a))\rangle + \langle \mu(a), [v_M, \rho(b)]\rangle \\ =& -\langle \mu(b), [v_M, \rho(a)]\rangle + \langle \mu(a), [v_M, \rho(b)]\rangle. \end{aligned}$$

So (3.3) becomes

$$\big\langle \mathcal{L}_{\rho(a)}\mu(b), v_M\big\rangle - \big\langle \mathcal{L}_{\rho(b)}\mu(a), v_M\big\rangle - \langle \mu(b), [v_M, \rho(a)]\rangle + \langle \mu(a), [v_M, \rho(b)]\rangle. \tag{3.4}$$

On the other hand

$$\big\langle \mathcal{L}_{\rho(a)} J_A(b), v\big\rangle = \mathcal{L}_{\rho(a)}\langle J_A(b), v\rangle = \big\langle \mathcal{L}_{\rho(a)}\mu(b), v_M\big\rangle + \langle \mu(b), [\rho(a), v_M]\rangle,$$

and swapping a and b gives

$$\big\langle \mathcal{L}_{\rho(b)} J_A(a), v\big\rangle = \big\langle \mathcal{L}_{\rho(b)}\mu(a), v_M\big\rangle + \langle \mu(a), [\rho(b), v_M]\rangle.$$

Comparing with (3.4), we see that (3.3) equals

$$\big\langle \mathcal{L}_{\rho(a)} J_A(b) - \mathcal{L}_{\rho(b)} J_A(a), v\big\rangle,$$

which immediately implies that (3.2) holds. $\square$

Suppose that A is integrable and $\mathcal{G}$ is the source-simply-connected Lie groupoid integrating it. We now make use of the Lie algebroid/groupoid version of Lie's second theorem (see e.g. the appendix of [16]). Since the G-action on A is by Lie-algebroid automorphisms, it has a natural lift to a G-action on $\mathcal{G}$ by Lie-groupoid automorphisms. Similarly, since $J_A : A \to \mathfrak{g}^*$ is a Lie algebroid morphism, it integrates to a Lie groupoid morphism

$$J : \mathcal{G} \to \mathfrak{g}^*, \tag{3.5}$$

where $\mathfrak{g}^*$ is now viewed as a groupoid over a point (i.e., an abelian group with respect to addition); i.e., J satisfies

$$J(gh) = J(g) + J(h), \tag{3.6}$$

for $(g,h) \in \mathcal{G}^{(2)}$. We also know from [5] (c.f. [1, 2, 3]) that the closed IM 2-form μ uniquely integrates to a closed 2-form ω in $\mathcal{G}$ which is *multiplicative*, in the sense that

$$m^*\omega = \mathrm{pr}_1^*\omega + \mathrm{pr}_2^*\omega, \tag{3.7}$$

where $\mathrm{pr}_1, \mathrm{pr}_2 : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ are the natural projections and $m : \mathcal{G}^{(2)} \to \mathcal{G}$ is the multiplication on $\mathcal{G}$. The relationship between ω and μ is given by

$$i_X\mu(a) = \omega(a, X) \tag{3.8}$$

for $a \in A$, $X \in TM$, where we view A and TM as subbundles of $T\mathcal{G}|_M$.

Note that the notion of being multiplicative, defined by condition (3.7), makes sense for differential forms of any degree; in particular, (3.6) says that J is a multiplicative ($\mathfrak{g}^*$-valued) function.

Proposition 3.2. *The map $J : \mathcal{G} \to \mathfrak{g}^*$ is G-equivariant and satisfies*

$$i_{v_\mathcal{G}}\omega = -\mathrm{d}\langle J, v\rangle, \quad \forall\, v \in \mathfrak{g}, \tag{3.9}$$

where $v_\mathcal{G}$ is the infinitesimal generator defined by v for the G-action on $\mathcal{G}$.

Proof. Let us verify the G-equivariance of $J : \mathcal{G} \to \mathfrak{g}^*$. Each $\sigma \in G$ defines an automorphism ϕ_σ of A as well as its global counterpart Φ_σ, which is an automorphism of $\mathcal{G}$. Then $(\mathrm{Ad}_\sigma^*)^{-1} \circ J_A \circ \phi_\sigma : A \to \mathfrak{g}^*$ is a Lie-algebroid morphism, whose global counterpart is $(\mathrm{Ad}_\sigma^*)^{-1} \circ J \circ \Phi_\sigma : \mathcal{G} \to \mathfrak{g}^*$. Since both J_{can} and μ are G-equivariant, so is J_A; i.e., $J_A = (\mathrm{Ad}_\sigma^*)^{-1} \circ J_A \circ \phi_\sigma$. It follows from the uniqueness of integration of morphisms that $J = (\mathrm{Ad}_\sigma^*)^{-1} \circ J \circ \Phi_\sigma$, i.e, J is G-equivariant.

The fact that G acts on $\mathcal{G}$ by Lie-groupoid automorphisms implies that the infinitesimal generators of this action are multiplicative vector fields (i.e., each $v_\mathcal{G} : \mathcal{G} \to T\mathcal{G}$ is a groupoid morphism, see e.g. [15]). It follows that the 1-form $i_{v_\mathcal{G}}\omega$ is multiplicative. Since J is also multiplicative (see (3.6)), so is $-\mathrm{d}\langle J, v\rangle$. To show that they are the same, it suffices (see e.g. [1, 2]) to check that these 1-forms agree on elements $a \in A_x = \ker(d\mathsf{s})|_x \subseteq T_x\mathcal{G}$, $x \in M$. Note that $v_\mathcal{G}(x) = v_M(x)$ for all $x \in M$. So

$$\omega_x(v_\mathcal{G}, a) = -\omega_x(a, v_M) = -i_{v_M}\mu(a).$$

On the other hand, using that $J_A = \mathrm{d}J|_A$,

$$-\langle \mathrm{d}J(a), v\rangle = -\langle J_A(a), v\rangle = -\langle \mu(a), v_M\rangle.$$

$\square$

As in [11, Thm. 3.3(ii)], one can check that there exists a map $j : M \to \mathfrak{g}^*$ such that $J = \mathsf{s}^* j - \mathsf{t}^* j$ if and only if $\langle \mu(a), u_M \rangle = -\mathrm{d}j^u(\rho(a))$, for all $u \in \mathfrak{g}$, $a \in A$, where $j^u(x) = \langle j(x), u \rangle$. The map j should be seen as a momentum map for the G-action on M preserving the closed IM 2-form $\mu : A \to T^*M$.

4. Reduction of closed IM 2-forms

We keep considering the set-up in *(i)*, *(ii)*, *(iii)* of Section 2.2, and $J_A : A \to \mathfrak{g}^*$ as in (3.1). Let us denote by $K \subseteq TM$ the distribution tangent to the G-orbits on M, and by $K^\circ \subseteq T^*M$ its annihilator.

Lemma 4.1. *If $J_A^{-1}(0)$ is a subbundle of A, then it is a Lie subalgebroid of A.*

Proof. The result follows from $\Gamma(J_A^{-1}(0))$ being closed under the Lie bracket on $\Gamma(A)$, as a consequence of (3.2). □

Remark 4.2. *If μ has constant rank, then $J_A^{-1}(0) \subseteq A$ is a subbundle if and only if* $\mathrm{Im}(\mu) \cap K^\circ$ *has constant rank. This follows from $J_{can}^{-1}(0) = K^\circ$ and $J_A = J_{can} \circ \mu$.*

Let us assume that $J_A^{-1}(0)$ is a subbundle of A and that the G-action on M is free and proper. The G-equivariance of J_A implies that the G-action on A keeps $J_A^{-1}(0)$ invariant; so we may consider the vector bundle (see (2.3))

$$A_{red} := J_A^{-1}(0)/G \to M/G.$$

The quotient map

$$p : M \to M/G \tag{4.1}$$

induces an isomorphism $p^* : C^\infty(M/G) \to C^\infty(M)^G$, where $C^\infty(M)^G$ denotes the space of G-invariant functions on M; also, $p^* A_{red} = J_A^{-1}(0)$ and the pull-back of sections

$$p^* : \Gamma(A_{red}) \to \Gamma(J_A^{-1}(0))$$

gives an identification between $\Gamma(A_{red})$ and $\Gamma(J_A^{-1}(0))^G$.

Proposition 4.3. *The vector bundle $A_{red} \to M/G$ inherits a natural Lie-algebroid structure, and the closed IM 2-form μ on A induces a closed IM 2-form μ_{red} on A_{red}.*

Proof. Since the G-action on $J_A^{-1}(0)$ is by Lie-algebroid automorphisms, the space $\Gamma(J_A^{-1}(0))^G$ is closed under the Lie bracket, and the identification

$$p^* : \Gamma(A_{red}) \xrightarrow{\sim} \Gamma(J_A^{-1}(0))^G$$

defines a Lie bracket $[\cdot,\cdot]_{red}$ on $\Gamma(A_{red})$ by $p^*[a,b]_{red} = [p^*a, p^*b]$. We define an anchor $\rho_{red} : A_{red} \to T(M/G)$ by the condition

$$\rho_{red}(a) = \mathrm{d}p(\rho(p^* a)), \tag{4.2}$$

for $a \in \Gamma(A_{red})$; note that this expression is well-defined (by the compatibility between ρ and the G-action, see (2.7)) and $C^\infty(M/G)$-linear. For a, $b \in \Gamma(A_{red})$ and $f \in C^\infty(M/G)$, the Leibniz identity follows from

$$\begin{aligned} p^*([a, fb]_{red}) &= [p^*a, (p^*f)p^*b] = (\mathcal{L}_{\rho(p^*a)} p^*f)p^*b + p^*f[p^*a, p^*b] \\ &= p^*((\mathcal{L}_{\mathrm{d}p(\rho(p^*a))} f)b + f[a,b]_{red}) = p^*((\mathcal{L}_{\rho_{red}(a)} f)b + f[a,b]_{red}). \end{aligned}$$

Since $J_A = J_{can} \circ \mu$, it is clear that $\mu : A \to T^*M$ restricts to $J_A^{-1}(0) \to J_{can}^{-1}(0) = K^\circ$; since it is G-equivariant, it descends to a bundle map

$$\mu_{red} : A_{red} \to T^*(M/G),$$

where we used the natural identification $K^\circ/G \cong T^*(M/G)$. It is a direct verification that the conditions (2.4), (2.5) for μ imply the same conditions for μ_{red}. □

5. Global reduction

We now discuss the global reduction associated with the data in (*i*), (*ii*), (*iii*) of Section 2.2. Let us assume that $J_A^{-1}(0) \subseteq A$ has constant rank, so it is a Lie subalgebroid of A, see Lemma 4.1. We assume that A is an integrable Lie algebroid and that $\mathcal{G}$ is the source-simply-connected Lie groupoid integrating it; let $J : \mathcal{G} \to \mathfrak{g}^*$ be the multiplicative function integrating J_A, as in Section 3. We say that $0 \in \mathfrak{g}^*$ is a *clean value* for J if $J^{-1}(0)$ is a submanifold and $\ker(\mathrm{d}J)|_g = T_g J^{-1}(0)$ for all $g \in J^{-1}(0)$.

Lemma 5.1. *If* 0 *is a clean value for* J*, then* $J^{-1}(0)$ *is a Lie subgroupoid of* $\mathcal{G}$ *over* M *whose Lie algebroid is* $J_A^{-1}(0)$*.*

Proof. Let $\tilde{\mathsf{s}}, \tilde{\mathsf{t}} : J^{-1}(0) \to M$ be the restrictions of s and t to $J^{-1}(0)$. In order to verify that $\tilde{\mathsf{s}}$ is a surjective submersion ($\tilde{\mathsf{t}}$ can be treated analogously), note that

$$\ker(\mathrm{d}\tilde{\mathsf{s}})_g = \ker(\mathrm{d}\mathsf{s})_g \cap T_g J^{-1}(0) = \ker(\mathrm{d}\mathsf{s})_g \cap \ker(\mathrm{d}J)_g, \quad g \in J^{-1}(0). \tag{5.1}$$

Since $M \subseteq J^{-1}(0)$, as a consequence of (3.6), and $\tilde{\mathsf{s}}|_M = \mathrm{Id}$, we see that $\mathrm{d}\tilde{\mathsf{s}}|_M$ is onto. It follows from (5.1) and $J_A = \mathrm{d}J|_A$ that

$$\ker(\mathrm{d}\tilde{\mathsf{s}})|_M = A \cap \ker(\mathrm{d}J)|_M = J_A^{-1}(0), \tag{5.2}$$

and a dimension count shows that

$$\mathrm{rank}(J_A^{-1}(0)) = \dim(J^{-1}(0)) - \dim(M). \tag{5.3}$$

For $g \in \mathcal{G}$, let $r_g : \mathsf{s}^{-1}(\mathsf{t}(g)) \to \mathsf{s}^{-1}(\mathsf{s}(g))$ be the associated right-translation map. Condition (3.6) implies that $(\mathrm{d}J)_g (\mathrm{d}r_g)_{\mathsf{t}(g)} a = (\mathrm{d}J)_{\mathsf{t}(g)}(a)$ for $a \in A_{\mathsf{t}(g)}$, which, along with (5.1), shows that, for $g \in J^{-1}(0)$,

$$\ker(\mathrm{d}\tilde{\mathsf{s}})_g = \ker(\mathrm{d}\mathsf{s})_g \cap \ker(\mathrm{d}J)_g = (\mathrm{d}r_g)_{\mathsf{t}(g)} A \cap \ker(\mathrm{d}J)_g = (\mathrm{d}r_g)_{\mathsf{t}(g)}(J_A^{-1}(0)).$$

It now follows from a dimension count using (5.3) that $\mathrm{rank}(\mathrm{d}\tilde{\mathsf{s}})_g = \dim(J^{-1}(0)) - \mathrm{rank}(J_A^{-1}(0)) = \dim(M)$, so $\tilde{\mathsf{s}}$ is a submersion.

A direct consequence of (3.6) is that $J^{-1}(0)$ is closed under the multiplication on $\mathcal{G}$, which endows $J^{-1}(0)$ with a Lie groupoid structure with source and target maps given by $\tilde{\mathsf{s}}$ and $\tilde{\mathsf{t}}$. As a result of (5.2), its Lie algebroid agrees with $J_A^{-1}(0)$. □

As in Section 4, we henceforth assume that the G-action on M is free and proper, which implies that the lifted G-action on $\mathcal{G}$ is also free and proper [11, Prop. 4.4]. If 0 is a clean value for J, so that $J^{-1}(0)$ is a submanifold (necessarily G-invariant by the equivariance of J), then the quotient

$$\mathcal{G}'_{red} := J^{-1}(0)/G$$

is a smooth manifold that naturally inherits a closed 2-form ω'_{red}; indeed, just as in usual Marsden-Weinstein reduction, condition (3.9) guarantees that the pull-back

of ω to $J^{-1}(0)$ is basic with respect to the G-action. Let A_{red} and μ_{red} be as in Prop. 4.3.

Proposition 5.2. *Suppose $0 \in \mathfrak{g}^*$ is a clean value for J. Then*

(1) *$\mathcal{G}'_{red}$ is a Lie groupoid over M/G such that the quotient map $J^{-1}(0) \to \mathcal{G}'_{red}$ is a groupoid homomorphism, and whose Lie algebroid is A_{red}.*

(2) *The closed 2-form ω'_{red} on $\mathcal{G}'_{red}$ is multiplicative and integrates the closed IM 2-form μ_{red} on A_{red}.*

Proof. The fact that $\mathcal{G}'_{red}$ inherits a groupoid structure, uniquely determined by the property that the quotient map

$$p : J^{-1}(0) \to \mathcal{G}'_{red}$$

is a homomorphism, can be directly checked using the freeness of the G-action (note that the restriction of this map to identity section $M \subseteq J^{-1}(0)$ agrees with the quotient map (4.1), hence the abuse of notation). Let $A(\mathcal{G}'_{red})$ be the Lie algebroid of $\mathcal{G}'_{red}$. The map p induces a surjective morphism of Lie algebroids

$$\mathrm{d}p|_{J_A^{-1}(0)} : J_A^{-1}(0) \to A(\mathcal{G}'_{red}), \tag{5.4}$$

covering the quotient map $M \to M/G$. We observe that $\ker(\mathrm{d}p) \cap \ker(\mathrm{d}\tilde{\mathsf{s}}) = \{0\}$ since $X \in \ker(\mathrm{d}p)$ if and only if $X = u_{\mathcal{G}}$ for some $u \in \mathfrak{g}$, and $\mathrm{d}\tilde{\mathsf{s}}(u_{\mathcal{G}}) = u_M = 0$ if and only if $u = 0$ by freeness. It follows that the kernel of (5.4) is trivial, so that (5.4) induces an identification between $J_A^{-1}(0)$ and the pull-back bundle $p^*A(\mathcal{G}'_{red})$, from where we obtain a natural identification between $A(\mathcal{G}'_{red})$ and $J_A^{-1}(0)/G = A_{red}$.

For the first assertion in (2), note that the pull-back $\iota^*\omega$ with respect to the inclusion $\iota : J^{-1}(0) \to \mathcal{G}'$ is a closed, multiplicative 2-form on $J^{-1}(0)$, and one can check that ω'_{red} is multiplicative from the equality $p^*\omega'_{red} = \iota^*\omega$ and the fact that p is a groupoid homomorphism.

To compare IM 2-forms, recall that $\mu_{red} : A_{red} \to T^*(M/G)$ is defined as the G-quotient of the restriction (see Prop. 4.3)

$$\mu|_{J_A^{-1}(0)} : J_A^{-1}(0) \to J_{can}^{-1}(0) = K^\circ.$$

Consider $\underline{a} \in A_{red}$ and $\underline{X} \in T(M/G)$ at a point $p(x) \in M/G$, and let $a \in J_A^{-1}(0)|_x$, $X \in T_xM$ be such that $\mathrm{d}p(X) = \underline{X}$ and $\mathrm{d}p(a) = \underline{a}$. Then

$$\omega'_{red}(\underline{a}, \underline{X}) = p^*\omega_{red}(a, X) = \iota^*\omega(a, X) = \mu(a)(X) = \mu_{red}(\underline{a})(\underline{X}),$$

which concludes the proof. □

When the 2-form ω on $\mathcal{G}$ is nondegenerate, then condition (3.9) and the freeness of the G-action on $\mathcal{G}$ guarantee that 0 is a regular value for J. In general, however, ω may be degenerate and (3.9) may give no information about the regularity of $J^{-1}(0)$. There is, nevertheless, an alternative route for constructing a "reduced" Lie groupoid over M/G that always works.

Let $\mathcal{G}_0$ be the source-simply-connected Lie groupoid integrating the Lie algebroid $J_A^{-1}(0)$ (since A is integrable, so is any Lie subalgebroid of A). Since $J_A^{-1}(0)$ is a Lie subalgebroid of A, $\mathcal{G}_0$ comes equipped with a groupoid homomorphism

$$\iota_0 : \mathcal{G}_0 \to \mathcal{G},$$

(which is an immersion but may fail to be injective, see e.g. [18]). The G-action on $J_A^{-1}(0)$ integrates to a G-action on $\mathcal{G}_0$ for which ι_0 is G-equivariant; moreover,

since the G-action on M is free and proper, so is the action on $\mathcal{G}_0$. The map $J\circ\iota_0 : \mathcal{G}_0 \to \mathfrak{g}^*$ is a multiplicative function whose infinitesimal counterpart $\mathrm{d}(J\circ\iota_0)|_{J_A^{-1}(0)} = J_A|_{J_A^{-1}(0)}$ vanishes. So $J\circ\iota_0 = 0$, i.e., $\iota_0(\mathcal{G}_0)\subseteq J^{-1}(0)$. As a result, one may use (3.9) to directly verify that $\iota_0^*\omega$ is a closed multiplicative 2-form on $\mathcal{G}_0$ which is basic with respect to the G-action. Hence

$$\mathcal{G}_{red} := \mathcal{G}_0/G$$

is a smooth manifold, and it inherits a closed 2-form ω_{red} from $\iota_0^*\omega$.

Proposition 5.3. *The following holds:*

1. *$\mathcal{G}_{red}$ is a Lie groupoid over M/G for which the quotient map $\mathcal{G}_0\to\mathcal{G}_{red}$ is a groupoid homomorphism, and whose Lie algebroid is A_{red}.*
2. *The closed 2-form ω_{red} on $\mathcal{G}_{red}$ is multiplicative, and it integrates the closed IM 2-form μ_{red} on A_{red}.*

The proof of Proposition 5.3 is similar to that of Proposition 5.2, but with no regularity assumptions on J.

As discussed in [11], the reduced groupoids $\mathcal{G}_{red}$ or $\mathcal{G}'_{red}$ generally do not agree with the source-simply-connected integration of A_{red}; [11, Thm. 4.11] shows how one can approach the problem of finding obstructions.

6. The case of Dirac structures

Let M be a manifold equipped with a Dirac structure L and a free and proper action of a Lie group G by Dirac diffeomorphisms. As explained in Section 2.3, this fits into the set-up in (i), (ii), (iii) as follows: A is the natural Lie algebroid defined on the vector-bundle $L\to M$ by (2.14) and $\mu=\mu_L$ is the closed IM 2-form defined in (2.15). As in Section 4, let $K\subseteq TM$ be the distribution tangent to the orbits of G on M, and let $K^\perp = TM\oplus K^\circ$ denote its orthogonal in $TM\oplus T^*M$ relative to the canonical pairing (2.12). Note that

$$L\cap K^\perp = \{(X,\alpha)\in L \mid \alpha\in K^\circ\} = \mu^{-1}(K^\circ) = J_A^{-1}(0). \tag{6.1}$$

Assuming that $L\cap K^\perp$ has constant rank, following Prop. 4.3, one may reduce A and μ to obtain A_{red} and μ_{red}, where

$$A_{red} = J_A^{-1}(0)/G = L\cap K^\perp/G \tag{6.2}$$

is a Lie algebroid over M/G and μ_{red} is a closed IM 2-form on A_{red}. On the other hand, one may consider the reduction of L in the sense of Dirac structures, recalled below, which produces a new Dirac structure on M/G. We will now compare these two possible reduction procedures.

6.1. Comparing reductions.

The lifted G-action on $\mathbb{T}M$ (see (2.16)) preserves the bundles K, $K^\perp$, and L, and we have a natural identification

$$\frac{K^\perp}{K}\Big/G = \frac{TM}{K}\oplus K^\circ\Big/G \cong T(M/G)\oplus T^*(M/G). \tag{6.3}$$

Assuming that the intersection $L\cap K^\perp$ has constant rank[2], the quotient

$$L_{quot} := \frac{L\cap K^\perp + K}{K}\Big/G \subset \frac{K^\perp}{K}\Big/G \tag{6.4}$$

[2]This is equivalent to $L\cap K$ having constant rank.

defines a Dirac structure on M/G, see [4, Sec. 4]. Equivalently, the Dirac structure (6.4) is the result of a push-forward by the quotient map $p : M \to M/G$:

$$(L_{quot})_{p(x)} = \{(\mathrm{d}p(X), \beta) \,|\, X \in T_xM,\ \beta \in T^*_{p(x)}(M/G),\ (X, \mathrm{d}p^*\beta) \in L_x\}. \tag{6.5}$$

Being a Dirac structure, L_{quot} carries a Lie-algebroid structure over M/G and is naturally equipped with the closed IM 2-form (c.f (2.15))

$$\mu_{quot} := \mathrm{pr}_{T^*}|_{L_{quot}} : L_{quot} \to T^*(M/G). \tag{6.6}$$

A direct comparison of (6.2) and (6.4) indicates that A_{red} and L_{quot} generally disagree. The following simple example illustrates this fact.

Example 6.1. *Consider the Dirac structure $L = TM$ and $\mu = \mu_L = 0$. One directly verifies that*

$$A_{red} = TM/G, \quad \mu_{red} = 0,$$

while $L_{quot} = T(M/G)$ and $\mu_{quot} = 0$, so the reductions do not match. In this case, ρ_{red} is the natural projection $TM/G \to T(M/G)$, so $\rho_{red}(A_{red}) = L_{quot}$.

We assume henceforth that $L \cap K^\perp$ has constant rank, so that both A_{red} and L_{quot} are well-defined. Let us consider the map

$$r : A_{red} \to T(M/G) \oplus T^*(M/G), \ \ a \mapsto (\rho_{red}(a), \mu_{red}(a)). \tag{6.7}$$

The following result relates A_{red}, μ_{red} and L_{quot}, μ_{quot} in general.

Lemma 6.2. *The image of the map r is L_{quot} and its kernel is $(K \cap L)/G$. Moreover, $r : A_{red} \to L_{quot}$ is a Lie-algebroid morphism and $\mu_{quot} \circ r = \mu_{red}$.*

Proof. From (6.2), we have

$$A_{red} = (K^\perp \cap L)/G = ((TM \oplus K^\circ) \cap L)/G.$$

Let us consider the natural vector-bundle maps

$$TM \to TM/G,\ X \mapsto \underline{X}, \quad K^\circ \to K^\circ/G \cong T^*(M/G),\ \alpha \mapsto \underline{\alpha},$$

as well as $\mathrm{d}p : TM \to T(M/G)$, all covering the quotient map $p : M \to M/G$. For $X \in T_xM$ and $\alpha \in (K^\circ)_x$, we have $\underline{\alpha}(\mathrm{d}p(X)) = \alpha(X)$, i.e., $(\mathrm{d}p)^*\underline{\alpha} = \alpha$. We can write

$$\begin{aligned}(A_{red})_{p(x)} &= \{(\underline{X}, \underline{\alpha}) \,|\, X \in T_xM, \alpha \in (K^\circ)_x,\ (X, \alpha) \in L_x\} \\ &= \{(\underline{X}, \underline{\alpha}) \,|\, X \in T_xM,\ (X, (\mathrm{d}p)^*\underline{\alpha}) \in L_x\}.\end{aligned}$$

With respect to this description of A_{red}, we have (c.f. (4.2))

$$\rho_{red}(\underline{X}, \underline{\alpha}) = \mathrm{d}p(X), \quad \mu_{red}(\underline{X}, \underline{\alpha}) = \underline{\alpha}. \tag{6.8}$$

Using (6.5), we see that the image of $(A_{red})_{p(x)}$ under (6.7) is

$$\{(\mathrm{d}p(X), \underline{\alpha}) \mid X \in T_xM,\ (X, (\mathrm{d}p)^*\underline{\alpha}) \in L_x\} = (L_{quot})_{p(x)}.$$

Recalling that $K = \ker(\mathrm{d}p)$, it is clear from (6.8) that

$$\ker(r) = \ker(\rho_{red}) \cap \ker(\mu_{red}) = (K \cap L)/G.$$

The fact that $r : A_{red} \to L_{quot}$ preserves Lie-algebroid structures follows from ρ_{red} preserving Lie brackets as well as conditions (2.4), (2.5) for μ_{red}. The compatibility $\mu_{quot} \circ r = \mu_{red}$ is immediate from (6.6). $\square$

The next result follows directly from Lemma 6.2.

Theorem 6.3. *The map r defines a Lie-algebroid isomorphism from A_{red} to L_{quot}, such that $\mu_{quot} \circ r = \mu_{red}$, if and only if $K \cap L = \{0\}$.*

We have the following consequences of Thm. 6.3:

(1) Whenever A_{red} and L_{quot} do not coincide, i.e., r fails to be an isomorphism, the Lie groupoid $(\mathcal{G}_{red}, \omega_{red})$ (resp. $(\mathcal{G}'_{red}, \omega'_{red})$, provided $J^{-1}(0)$ is a smooth submanifold) constructed in Section 5 is *not* a presymplectic groupoid for L_{quot}; by Thm. 6.3, this happens if and only if $K \cap L = \{0\}$. In general, $(\mathcal{G}_{red}, \omega_{red})$ (resp. $(\mathcal{G}'_{red}, \omega'_{red})$) is an *over-presymplectic groupoid* in the sense of [5, Def. 4.5]; in particular, the target map $\mathtt{t} : (\mathcal{G}_{red}, \omega_{red}) \to (M/G, L_{quot})$ is always a forward Dirac map.

(2) If L is the graph of a Poisson structure π on M, then it immediately follows that $K \cap L = \{0\}$ and $L \cap K^{\perp}$ has constant rank. In this case, L_{quot} is just the graph of the natural Poisson structure π_{quot} on M/G, uniquely determined by $p : M \to M/G$ being a Poisson map. By Thm. 6.3, A_{red} is identified with L_{quot} via r, and $(\mathcal{G}_{red}, \omega_{red})$ is a symplectic groupoid for π_{quot}; since 0 is automatically a regular value for J in this case, one may also consider the (possibly different) symplectic groupoid $(\mathcal{G}'_{red}, \omega'_{red})$. This situation is fully treated and further developed in [11].

(3) Let us suppose that the Dirac structure L on M is such that L_{quot} is given by the graph of a Poisson structure π_{quot}. In this case, we note that r defines an isomorphism $A_{red} \cong L_{quot}$ if and only if L is itself the graph of a Poisson structure, i.e., $L \cap TM = \{0\}$; indeed, it follows from (6.5) that

$$L_{quot} \cap T(M/G) = \mathrm{d}p(L \cap TM),$$

so $L_{quot} \cap T(M/G) = \{0\}$ if and only if $L \cap TM \subseteq K = \ker(\mathrm{d}p)$. But since $L \cap K = \{0\}$, it follows that $L \cap TM = \{0\}$.

As a consequence, the reduced groupoid $(\mathcal{G}_{red}, \omega_{red})$ (resp. $(\mathcal{G}'_{red}, \omega'_{red})$) is *not* a symplectic groupoid for π_{quot} unless L is the graph of a Poisson structure to begin with. This observation indicates that the reduction in [11, Thm. 4.21] for non-free Poisson actions may not generally yield symplectic groupoids on each strata (just *over*-symplectic [5, Def. 4.5]).

6.2. **A non-integrable quotient.** Let us keep considering a manifold M equipped with a Dirac structure L and carrying a free and proper G-action by Dirac diffeomorphisms. We saw in Thm. 6.3 that A_{red} and L_{quot} do not coincide in general. We now illustrate how different these Lie algebroids can be concerning integrability.

It is a direct consequence of Prop. 5.3 that, if L is integrable as a Lie algebroid, then so is A_{red}. It may happen, however, that the quotient Dirac structure L_{quot} is *not* integrable as a Lie algebroid. Note that this feature is not present when L is defined by a Poisson structure, as in this case L_{quot} necessarily coincides with A_{red}; i.e, the quotient of an integrable Poisson structure by a free and proper G-action by Poisson diffeomorphisms is always integrable (see [11, Prop. 4.6]). We will verify how this picture changes in the realm of Dirac structures through a concrete example.

Recall that $P = S^2 \times \mathbb{R}$, with coordinates (x, t), may be equipped with a Poisson structure π_P that does not admit an integration to a symplectic groupoid [20] (see also [10]). This nonintegrable Poisson structure is characterized by the fact that its symplectic leaves are the spheres obtained by constant values of t endowed with $(1 + t^2)$ times the usual symplectic structure of S^2.

Proposition 6.4. *There exists a presymplectic manifold (M, ω) carrying a free and proper G-action preserving ω for which $P = M/G$ and the quotient map $M \to P$ is a forward Dirac map; i.e., if L is the graph of ω, then L_{quot} is the graph of π_P.*

Observe that whenever L is the graph of a presymplectic structure, it is automatically integrable as a Lie algebroid, as the projection $\mathrm{pr}_T|_L : L \to TM$ is a Lie-algebroid isomorphism, so L is integrated by the fundamental groupoid of M, see e.g. [7]. As already mentioned, since π_P is nonintegrable, it is impossible to replace ω in Prop. 6.4 by a symplectic form (or any other integrable Poisson structure), so ω must be degenerate.

The proof of Prop. 6.4 is a direct consequence of the construction in [5, Example 6.8], that we recall for completeness.

Let $\mathfrak{g}$ be a Lie algebra endowed with an Ad-invariant, symmetric and nondegenerate bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. We use B to identify $\mathfrak{g}^* \cong \mathfrak{g}$, in such a way that the adjoint and co-adjoint actions are intertwined. We consider the function

$$C : \mathfrak{g}^* \to \mathbb{R}, \ \ \xi \mapsto \frac{1}{2} B(\xi, \xi).$$

This function is a Casimir for the linear Poisson structure $\pi_{\mathfrak{g}^*}$ on $\mathfrak{g}^*$, as a result of the Ad-invariance of B. It follows that $C^{-1}(\lambda)$ is a Poisson submanifold of $\mathfrak{g}^*$ for any $\lambda \in \mathbb{R}$, i.e., $\pi_{\mathfrak{g}^*}$ restricts to a Poisson structure π_λ on $C^{-1}(\lambda)$ for which the inclusion

$$j_\lambda : C^{-1}(\lambda) \to \mathfrak{g}^*$$

is a Poisson map.

Let G be a connected Lie group whose Lie algebra is $\mathfrak{g}$, and let $G \times \mathfrak{g}^*$ be endowed with the symplectic structure Ω coming from the identification $G \times \mathfrak{g}^* \cong T^*G$ via right translations, so that the projection on the second factor,

$$p : G \times \mathfrak{g}^* \to \mathfrak{g}^*,$$

is a Poisson map. The submanifold $M_\lambda = (C \circ p)^{-1}(\lambda) = G \times C^{-1}(\lambda)$ of $G \times \mathfrak{g}^*$ is equipped with a closed 2-form ω_λ, given by the pull-back of Ω by the inclusion

$$\iota_\lambda : M_\lambda \hookrightarrow G \times \mathfrak{g}^*,$$

and carries a free and proper G-action (by right multiplication on the first factor) so that $M_\lambda/G = C^{-1}(\lambda)$. We denote the quotient map by $p_\lambda : M_\lambda \to C^{-1}(\lambda)$, so that

$$j_\lambda \circ p_\lambda = p \circ \iota_\lambda.$$

Let L be the Dirac structure on M_λ given by the graph of ω_λ. Consider the distribution $K = \ker(p_\lambda) = TG \subseteq TM_\lambda$. Since $M_\lambda \subset G \times \mathfrak{g}^*$, we may view K in $T(G \times \mathfrak{g}^*)$ and consider its symplectic orthogonal $K^\Omega \subseteq T(G \times \mathfrak{g}^*)$.

Lemma 6.5. *$K^\Omega \subseteq TM_\lambda$ and $L \cap K^\perp = \{(X, i_X\omega_\lambda) \,|\, X \in K^\Omega\}$. In particular, $L \cap K^\perp$ has constant rank.*

Proof. The distribution K^Ω is spanned by hamiltonian vector fields X_{p^*f} of functions of the form p^*f, for $f \in C^\infty(\mathfrak{g}^*)$. Since p is a Poisson map, we have $\mathrm{d}p(X_{p^*f}) = X_f$. It follows that

$$\mathrm{d}C(\mathrm{d}p(X_{p^*f})) = \mathrm{d}C(X_f) = 0,$$

for C is a Casimir. Hence $K^\Omega \subseteq \ker(\mathrm{d}(C \circ p)) = TM_\lambda$.

Since $K^\perp = TM_\lambda \oplus K^\circ$, $L \cap K^\perp = \{(X, i_X\omega_\lambda) \,|\, X \in TM_\lambda,\, i_X\omega_\lambda \in K^\circ\}$. But one can directly check that $i_X\omega_\lambda \in K^\circ$ if and only if $X \in TM_\lambda \cap K^\Omega = K^\Omega$. □

Let L_{quot} be the reduction of L as in (6.4), (6.5).

Lemma 6.6. *L_{quot} is the graph of π_λ.*

Proof. For each $\xi \in C^{-1}(\lambda)$ and $\sigma \in G$, we have

$$(L_{quot})_\xi = \{(\mathrm{d}p_\lambda(X), (\mathrm{d}f)_\xi) \,|\, X \in T_{(\sigma,\xi)}M_\lambda,\ f \in C^\infty(C^{-1}(\lambda)),\ \text{ and }\ i_X\omega_\lambda = \mathrm{d}p_\lambda^* f\}.$$

We must compare it with the graph of π_λ, given at $\xi \in C^{-1}(\lambda)$ by

$$\{((\pi_\lambda)^\sharp((\mathrm{d}f)_\xi), (\mathrm{d}f)_\xi) \,|\, f \in C^\infty(C^{-1}(\lambda))\}.$$

To conclude that they coincide, it is enough to check that L_{quot} is contained in the graph of π_λ, as both vector-bundles have the same rank. In other words, it suffices to show that for $X \in T_{(\sigma,\xi)}M_\lambda$ and $f \in C^\infty(C^{-1}(\lambda))$ such that $i_X\omega_\lambda = \mathrm{d}p_\lambda^* f$ then

$$\mathrm{d}p_\lambda(X) = (\pi_\lambda)^\sharp((\mathrm{d}f)_\xi). \tag{6.9}$$

Let $\hat{f} \in C^\infty(\mathfrak{g}^*)$ be any extension of f, so that $f = \hat{f} \circ j_\lambda$. Then

$$\iota_\lambda^*(i_X\Omega) = i_X\iota_\lambda^*\Omega = i_X\omega_\lambda = \mathrm{d}p_\lambda^* j_\lambda^* \hat{f} = \iota_\lambda^* \mathrm{d}p^* \hat{f},$$

which implies that

$$i_X\Omega = \mathrm{d}p^*\hat{f} + k\mathrm{d}(p^*C) \tag{6.10}$$

for some $k \in \mathbb{R}$. Denoting by π_Ω the Poisson structure defined by Ω, the fact that $p : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ is a Poisson map implies that

$$\mathrm{d}p(\pi_\Omega^\sharp \mathrm{d}p^* C) = \pi_{\mathfrak{g}^*}^\sharp(\mathrm{d}C) = 0,$$

since C is a Casimir. It follows from (6.10) that $X = \pi_\Omega^\sharp(\mathrm{d}p^*\hat{f} + k\mathrm{d}(p^*C))$, so

$$\mathrm{d}p(X) = \mathrm{d}p(\pi_\Omega^\sharp(\mathrm{d}p^*\hat{f} + k\mathrm{d}(p^*C))) = \mathrm{d}p(\pi_\Omega^\sharp(\mathrm{d}p^*\hat{f})) = \pi_{\mathfrak{g}^*}^\sharp(\mathrm{d}\hat{f}).$$

We conclude that (6.9) holds as a direct consequence of j_λ being a Poisson map and X being tangent to M_λ. □

Remark 6.7. *It follows from Lemma 6.5 that $L \cap K^\perp$ is isomorphic to the distribution $K^\Omega \subseteq TM_\lambda$, which can be identified with the action Lie algebroid $\mathfrak{g} \ltimes M_\lambda$ arising from the diagonal action of G on M_λ by left multiplication on the first factor and the coadjoint action on the second. By* (6.2)*, one can check that $A_{red} = K^\Omega/G$ is identified with the action Lie algebroid $\mathfrak{g} \ltimes C^{-1}(\lambda)$ (relative to the coadjoint action). If G is simply connected, the reduced groupoid $\mathcal{G}_{red}$ agrees with the action Lie groupoid $G \ltimes C^{-1}(\lambda)$, and ω_{red} coincides with ω_λ.*

For the proof of Proposition 6.4, consider the Lie algebras $\mathfrak{su}(2)$ and $\mathbb{R}$, equipped with their canonical bilinear forms $B_{\mathfrak{su}(2)}$ and $B_\mathbb{R}$. With the usual identification $\mathfrak{su}(2) \cong \mathbb{R}^3$, $B_{\mathfrak{su}(2)}$ agrees with the euclidean inner product. We let $\mathfrak{g}$ be the Lie-algebra direct sum $\mathfrak{su}(2) \oplus \mathbb{R}$, equipped with the bilinear form $B = B_{\mathfrak{su}(2)} - B_\mathbb{R}$. Then

$$C^{-1}(1/2) = \{(\xi, t) \in \mathbb{R}^3 \times \mathbb{R} \,|\, \langle \xi, \xi \rangle = 1 + t^2\}$$

can be identified with $P = S^2 \times \mathbb{R}$ in such a way that $\pi_{1/2}$ agrees with π_P. Lemma 6.6 shows that $M = \mathrm{SU}(2) \times \mathbb{R} \times C^{-1}(1/2)$ can be equipped with a presymplectic form that pushes forward to π_P. This concludes the proof of Proposition 6.4.

7. Relation with the path-space model

Given an integrable Lie algebroid A over M, there is a canonical model for the source-simply-connected Lie groupoid integrating it in terms of paths in A [9] (see also [6, 19]): one considers the space $\widetilde{P}(A)$ of all C^1 paths $a: I \to A$ from the interval $I = [0,1]$ into A so that the projected path $q_A \circ a : I \to M$, where $q_A : A \to M$ is the vector-bundle projection, is of class C^2. This space has the structure of a Banach manifold. We let $P(A)$ be the submanifold of A-*paths*, equipped with the equivalence relation given by A-*homotopy*, denoted by $\sim$, see [9]. Then the quotient

$$\mathcal{G}(A) := P(A)/\sim$$

is a source-simply-connected Lie groupoid integrating A. This explicit model was used in the study of Poisson actions in [11]. We now briefly explain how it relates to our approach via closed IM-forms.

Let us consider the set-up described in (i), (ii), (iii) of Section 2.2, let $J_A : A \to T^*M$ be as in (3.1) and $J : \mathcal{G}(A) \to \mathfrak{g}^*$ be its global counterpart, see Prop. 3.2.

Proposition 7.1. *The map $J : \mathcal{G}(A) \to \mathfrak{g}^*$ is given by*

$$\langle J([a]), u\rangle = \int_I \big\langle \mu(a(t)), u_M|_{q_A(a(t))}\big\rangle dt, \tag{7.1}$$

where $u \in \mathfrak{g}$, u_M is the associated infinitesimal generator and $a \in P(A)$.

When A is defined by a Dirac structure $L \subset TM \oplus T^*M$, one replaces μ by pr_{T^*} in the formula (7.1); if L is the graph of a Poisson structure, one recovers the formula for J in [11, Thm. 3.3].

Proposition 7.1 follows from a more general observation. Let $A_1 \to M_1$ and $A_2 \to M_2$ be Lie algebroids and $\psi : A_1 \to A_2$ be a vector-bundle map. We denote by $\widetilde{\psi} : \widetilde{P}(A_1) \to \widetilde{P}(A_2)$ the induced map on paths.

Lemma 7.2. *$\widetilde{\psi}$ takes A_1-paths to A_2-paths (i.e., $\widetilde{\psi}(P(A_1)) \subseteq P(A_2)$) preserving A-homotopy (i.e., $a \sim_{A_1} a'$ implies that $\widetilde{\psi}(a) \sim_{A_2} \widetilde{\psi}(a')$ for all a, $a' \in P(A_1)$) if and only if ψ is a Lie-algebroid morphism.*

If ψ is a Lie-algebroid morphism, it follows that the map $\widetilde{\psi}|_{P(A_1)} : P(A_1) \to P(A_2)$ descends to a map $\mathcal{G}(A_1) \to \mathcal{G}(A_2)$, which is the groupoid morphism integrating ψ.

Proof. (of Prop. 7.1) For a vector space V, thought of as a trivial Lie algebra (or a trivial Lie algebroid over a point), $\widetilde{P}(V) = P(V)$ and the quotient map $P(V) \to \mathcal{G}(V) = V$ is given by $a(t) \mapsto \int_I a(t)dt$.

Considering the Lie-algebroid morphism $J_A = J_{can} \circ \mu : A \to \mathfrak{g}^*$, it follows that the composition of $\widetilde{J_A}|_{P(A)} : P(A) \to P(\mathfrak{g}^*)$ with $P(\mathfrak{g}^*) \to \mathcal{G}(\mathfrak{g}^*) = \mathfrak{g}^*$ is

$$a(t) \mapsto \int_I J_A(a(t))dt.$$

By Lemma 7.2, the map $J : \mathcal{G}(A) \to \mathcal{G}(\mathfrak{g}^*) = \mathfrak{g}^*$ is given by $J([a(t)]) = \int_I J_A(a(t))dt$, hence $\langle J([a(t)]), u\rangle = \int_I \langle J_A(a(t)), u\rangle dt = \int_I \big\langle \mu(a(t)), u_M|_{q_A(a(t))}\big\rangle$. □

One can also use Lemma 7.2 and [2] to generalize formula (7.1) to describe multiplicative k-forms in terms of the path-space model.

References

[1] Arias Abad, C., Crainic, M., The Weil algebra and the Van Est isomorphism. Arxiv: 0901.0322, to appear in Ann. Inst. Fourier.

[2] Bursztyn, H., Cabrera, A., Multiplicative forms at the infinitesimal level. Arxiv: 1001.0534, to appear in *Math. Annalen*.

[3] Bursztyn, H., Cabrera, A., Ortiz, C., Linear and multiplicative 2-forms. *Lett. Math. Phys.*, **90** (2009), 59–83.

[4] Bursztyn, H., Cavalcanti, G., Gualtieri, M., Reduction of Courant algebroids and generalized complex structures. *Advances in Math.* **211** (2007), 726–765.

[5] Bursztyn, H., Crainic, M., Weinstein, A., Zhu, C., Integration of twisted Dirac brackets, *Duke Math. J.* **123** (2004), 549-607.

[6] Cattaneo, A., Felder, G., *Poisson sigma models and symplectic groupoids.* Quantization of singular symplectic quotients, 61–93, Progr. Math., **198**, Birkhauser, Basel, 2001.

[7] Coste, A., Dazord, P., Weinstein, A., *Groupoïdes symplectiques.* Publications du Département de Mathématiques. Nouvelle Série. A, Vol. 2, i–ii, 1–62, Publ. Dép. Math. Nouvelle Sér. A, 87-2, Univ. Claude-Bernard, Lyon, 1987.

[8] Courant, T., Dirac manifolds, *Trans. Amer. Math. Soc.* **319** (1990), 631-661.

[9] Crainic, M., Fernandes, R., Integrability of Lie brackets. *Ann. of Math.* **157** (2003), 575–620.

[10] Crainic, M., Fernandes, R., Integrability of Poisson brackets. *J. Differential Geom.* **66** (2004), 71-137.

[11] Fernandes, R., Ortega, J.-P., Ratiu, T., The momentum map in Poisson geometry. *Amer. J. of Math.* **131** (2009), 1261-1310

[12] Jotz, M., Ratiu, T., Induced Dirac structures on isotropy type manifolds. *Transform. Groups* **16** (2011), 175–191.

[13] Jotz, M., Ratiu, T., Sniatycki, J., Singular reduction of Dirac structures. *Trans. Amer. Math. Soc.* **363** (2011), 2967–3013.

[14] Lu, J.-H,, *private communication.*

[15] Mackenzie, K., Xu, P., Classical lifting processes and multiplicative vector fields. *Quart. J. Math.* **49** (1998), 59-85.

[16] Mackenzie, K., Xu, P., Integration of Lie bialgebroids. *Topology* **39** (2000), 445–467.

[17] Mikami, K., Weinstein, A., Moments and reduction for symplectic groupoid actions. *Publ. RIMS, Kyoto Univ.* **24** (1988), 121-140.

[18] Moerdijk, I., Mrcun, J., On the integrability of Lie subalgebroids. *Adv. Math.* **204** (2006), 101115.

[19] Ševera, P., Some title containing the words "homotopy" and "symplectic", e.g. this one. *Travaux mathématiques.* Fasc. XVI (2005), 121–137.

[20] Weinstein, A., Symplectic groupoids and Poisson manifolds. *Bull. Amer. Math. Soc. (N.S.)* **16** (1987), 101–104.

Instituto de Matemática Pura e Aplicada, Estrada Dona Castorina 110, Rio de Janeiro, 22460-320, Brasil

E-mail address: `henrique@impa.br, cabrera@impa.br`

Departamento de Matemática Aplicada, Instituto de Matemática, Universidade Federal do Rio de Janeiro, CEP 21941-909, Rio de Janeiro - RJ, Brazil

E-mail address: `cabrera@impa.br`